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$W_{1+\infty}$ as a discretization of Virasoro algebra

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Abstract. It is shown that a realization of $W_{1+\infty}$ algebra in the KP hierarchy is equivalent to a difference operator realization of the simplest subalgebra of a q-discretized Virasoro algebra (D-$\mathrm{Vir}_c^{(+,0)}$). To this end, we first introduce the non-commutative flow of the KP hierarchy and see the *additional* flow corresponds to the element of the $W_{1+\infty}$ algebra. It is then shown that the new flow constructed by use of the additional one can be recognized as an action of D-$\mathrm{Vir}_c^{(+,0)}$ in the Fock representation of the KP hierarchy.

1. Introduction

In research on integrable systems, discretization of variables provides important information about the systems. Stability under discretization characterizes a certain class of integrable systems which are completely integrable. For example, every soliton equation of the KP hierarchy [1, 2] has a discrete analogue which is common to all equations. Namely, a single bilinear difference equation of Hirota reproduces every soliton equation of the KP hierarchy by taking certain continuous limits of the variables [3, 4]. This remarkable property of soliton-type equations should be compared with a generic case in which discretization of variables in general makes nonlinear equations create chaos [5]. This means that there exist some symmetries which preserve integrability under discretization of variables.

Discretization of differential operators also plays a key role in q-deformed conformal field theories. The q-deformed Knizhnik–Zamolodchikov (q-KZ) equation of Frenkel and Reshetikhin [6] has been formulated by considering the representation theory of $U_q(\widehat{sl_2})$, and is essentially realized through discretization of variables of the original KZ equation.

We are interested in relations which remain true irrespective of whether they are continuous or discrete. Such relations will not only characterize completely integrable systems but also add to the understanding of the boundary between deterministic and non-deterministic nonlinear equations.

In order to claim this idea, in a series of papers [7] we proposed a deformation of the Virasoro algebra which was realized by a q-discretization of differential operators, and called it D-Virasoro. This algebra was shown to admit both fermionic and bosonic free field representations [16]. It was constructed such that a realization of the Virasoro algebra was reproduced in the continuous limit ($q \to 1$). Although it was derived in as general a form as possible, there remains the problem of clarifying its relation with other known symmetries. An indication can be seen if we consider the above algebra from the Moyal-algebraic point of view. The Moyal bracket algebra, which was provided in [19–21], has a

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large structure in the sense that this algebra is recognized as a deformation of the Poisson algebra. D-Virasoro is simply a series of subalgebras of it.

In this paper we confirm, by specific realization, that the simplest D-Virasoro subalgebra is quite simply the $W_{1+\infty}$ subalgebra of the Moyal algebra [11]. This result provides a new interpretation of the $W_{1+\infty}$ algebra. Namely, it enables us to understand that the $W_{1+\infty}$ algebra emerges as a result of a proper discretization of the Virasoro algebra.

In section 2, we introduce the non-commutative symmetry flow of the KP hierarchy which is found in [8] and generalized in [9, 10]. They will be shown to be equivalent to a realization of the $W_{1+\infty}$ algebra. In section 3, we first give a brief review of the discretization of the Virasoro algebra (D-Virasoro). From the fact that one of the D-Virasoro subalgebras ($D\text{-Vir}_{\rm c}^{(+,0)}$) has the free fermion field representation, we discuss the relation between $D\text{-Vir}_{\rm c}^{(+,0)}$ and $W_{1+\infty}$ through the Fock representation of the KP hierarchy.

2. $W_{1+\infty}$ in the KP hierarchy

The KP hierarchy is given by use of the pseudo-differential operators $L = K\partial K^{-1}$ ($K = 1 + \sum_{j>0} a_j(x,t)\partial^{-j}$) as

$$\frac{\partial K}{\partial t_r} = B_r K - K\partial^r \qquad B_r = (K\partial^r K^{-1})_+ \tag{2.1}$$

where $(\cdots)_+$ denotes the differential operator part. The action ∂_r can be understood as the vector field which determines the dynamical flow (KP flow) not only on the pseudo-differential operator ring but also on the universal Grassmann manifold (UGM), because of the isomorphism between them. From (2.1), we immediately get $[\partial_r, \partial_{r'}] = 0$. Hence the KP flow is simply the infinite commutative symmetry.

In contrast to the KP flow, we can also consider the non-commutative flow, i.e. flow which does not commute with each other, but does commute with the KP flow [8]. First recall that (2.1) is equivalent to the following system of linear equations:

$$L\phi = z\phi \qquad \partial_r\phi = L_+^r\phi \tag{2.2}$$

where the wavefunction $\phi(z,t)$ is given by

$$\phi(z,t) = K\exp\sum_r t_r z^r \equiv K\mathrm{e}^{\xi}\,. \tag{2.3}$$

Similarly, the derivation ∂_z to the wavefunction is written in terms of the pseudo-differential operators as

$$\frac{\partial\phi}{\partial z} = K\left(\sum_{r=1}^{\infty} r t_r \partial^{r-1}\right)K^{-1}K\mathrm{e}^{\xi} \equiv M\phi\,. \tag{2.4}$$

Then we get $z^m\partial_z^l\,\phi = M^l L^m\phi$ ($m\in\mathbb{Z},\ l\in\mathbb{Z}_{\geqslant 0}$). It enables us to consider the vector field as

$$\partial_{ml}\ :\ \partial_{ml}K = -(M^l L^m)_- K\,. \tag{2.5}$$

Moreover, between $z^m\partial_z^l$ and ∂_{ml} there is a Lie algebra isomorphism $z^m\partial_z^l \mapsto \partial_{ml}$. Therefore ∂_{ml} yields the following relations:

$$[\partial_{ml}, \partial_r] = 0$$

$$[\partial_{ml}, \partial_{m'l'}] = \sum_{j=1}^{\infty}\left\{\binom{m}{j}\binom{l'}{j} - \binom{m'}{j}\binom{l}{j}\right\} j!\ \partial_{m+m'-j,\ l+l'-j}\,. \tag{2.6}$$

In general, the operators spanned by the differential operators $\{z^m \partial_z^l; m \in \mathbb{Z}, l \in \mathbb{Z}_{\geqslant 0}\}$ realize an infinite-dimensional Lie algebra, $w_{1+\infty}$. After all, the vector fields ∂_{mk} are simply the $w_{1+\infty}$ flow.

The central extended algebra of $w_{1+\infty}$, $W_{1+\infty}$ [11], is related to the infinitesimal Bäcklund transformation of the τ function of the KP hierarchy

$$\tau \to \tau + \varepsilon X(z, \zeta)\ \tau . \tag{2.7}$$

Here $X(z, \zeta)$ is the vertex operator

$$\begin{aligned} X(z, \zeta) &= \frac{: V(z) V^*(\zeta) : -1}{z - \zeta} \\ V(z) &= \exp\left(\sum_{j=1}^{\infty} z^j t_j\right) \exp\left(-\sum_{j=1}^{\infty} \frac{1}{j} z^{-j} \partial_j\right) \\ V^*(z) &= \exp\left(-\sum_{j=1}^{\infty} z^j t_j\right) \exp\left(\sum_{j=1}^{\infty} \frac{1}{j} z^{-j} \partial_j\right) . \end{aligned} \tag{2.8}$$

From equation (2.5) the action of ∂_{mk} on the wavefunction becomes

$$\partial_{ml}\phi = -(M^l L^m)_- \phi . \tag{2.9}$$

Rewriting ϕ by use of the vertex operator and the τ function,

$$\phi(z, t) = \frac{V(z)\ \tau}{\tau} = \frac{\tau(t_1 - \frac{1}{z}, t_2 - \frac{1}{z^2}, \ldots)}{\tau(t_1, t_2, \ldots)}\ \mathrm{e}^{\xi(z,t)} \tag{2.10}$$

then we get

$$\partial_{ml}\ \tau = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\ \left(z^m \partial_z^l\ V(z)\right)\ V^*(z)\ \tau \tag{2.11}$$

(see the appendix) [12, 13]. If we define $\hat{\partial}_{ml} \in W_{1+\infty}$, we also obtain

$$\hat{\partial}_{ml}\ \tau = \oint \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\ \ z^m \partial_z^l\ X(z, \zeta)\Big|_{z=\zeta}\ \tau . \tag{2.12}$$

Several authors also discussed the $W_{1+\infty}$ structure by using this non-commuting flow [14, 15]. In the following section, we try to clarify that the $W_{1+\infty}$ algebra can be understood as a deformation of the Virasoro algebra.

3. The relation between $W_{1+\infty}$ and D-Virasoro

Let us consider a kind of q-difference operator

$$\mathcal{L}_m^{(n)}(z) = z^m\ \frac{q^{n(z\partial_z + m/2)} - q^{-n(z\partial_z + m/2)}}{q^n - q^{-n}} . \tag{3.1}$$

These operators $\{\mathcal{L}_m^{(n)}; m \in \mathbb{Z}, n \in \mathbb{Z}_{>0}\}$ satisfy the infinite-dimensional Lie algebra which obeys the following commutation relations:

$$\left[\mathcal{L}_m^{(n)}, \mathcal{L}_{m'}^{(n')}\right] = C\left(\begin{smallmatrix} m & n \\ m' & n' \end{smallmatrix}\right)\ \mathcal{L}_{m+m'}^{(n+n')} + C\left(\begin{smallmatrix} m & n \\ m' & -n' \end{smallmatrix}\right)\ \mathcal{L}_{m+m'}^{(n-n')} \tag{3.2}$$

where

$$C\left(\begin{smallmatrix} m & n \\ m' & n' \end{smallmatrix}\right) = \frac{\left[\frac{1}{2}(nm' - n'm)\right][n + n']}{[n][n']} \qquad [x] = \frac{q^x - q^{-x}}{q - q^{-1}} .$$

In the limit of $q \to 1$, $\mathcal{L}_m^{(n)} \to l_m = z^m(z\partial_z + m/2)$. l_m is simply a realization of the Virasoro algebra

$$[\ l_m\ ,\ l_{m'}] = (m' - m)\ l_{m+m'}. \tag{3.3}$$

Then we can regard (3.2) as a deformation of the Virasoro algebra, i.e. if we 'discretize' the differential operator in the above realization of Virasoro, then we get a realization of a larger algebra. In this limited sense, we choose to call $\{\mathcal{L}_m^{(n)}\}$ a q-discretized Virasoro algebra, or D-Virasoro for short. D-Virasoro has a more general form (D-Vir) [7, 16].

$$\begin{aligned}
\left[\mathcal{L}_m^{(n,r;\pm)}, \mathcal{L}_{m'}^{(n',r;\pm)}\right] &= C\binom{m\ \ n+r}{m'\ \ n'+r}_{\pm}\ \mathcal{L}_{m+m'}^{(n+n'+r,r;\pm)} + C\binom{m\ \ n-r}{m'\ \ n'-r}_{\pm}\ \mathcal{L}_{m+m'}^{(n+n'-r,r;\pm)} \\
&\quad + C\binom{m\ \ n+r}{m'\ \ -n'+r}_{\pm}\ \mathcal{L}_{m+m'}^{(n-n'+r,r;\pm)} + C\binom{m\ \ n-r}{m'\ \ -n'-r}_{\pm}\ \mathcal{L}_{m+m'}^{(n-n'-r,r;\pm)} \\
C\binom{m\ \ n+r}{m'\ \ n'+r}_{\pm} &\equiv \frac{\left[\frac{1}{2}((n+r)m' - (n'+r)m)\right]_-\ \ [n+n'+r]_{\mp}}{2\ [n]_{\mp}\ [n']_{\mp}\ [r]_{\pm}} \\
[x]_+ &\equiv \frac{q^x + q^{-x}}{2} \qquad [x]_- \equiv [x]\,.
\end{aligned} \tag{3.4}$$

(The double signs on both sides correspond to each other.) Equation (3.2) is a subalgebra of D-Vir (set $r = 0$ in the (+)-type). Then we denote it D-Vir$^{(+,0)}$. We remark that D-Vir$^{(+,0)}$ is formally identified with the first ($a = 0$) subalgebra of the Moyal algebra in [20, section V]†. The central extension of D-Vir$^{(+,0)}$ is also obtained (D-Vir$_{\rm c}^{(+,0)}$). It has the free fermion field representation

$$\hat{\mathcal{L}}_m^{(n)} = \frac{1}{2}\sum_{p\in\mathbb{Z}} A_{p,m-p}^{(n)}\ \ :\psi_{m-p}\ \psi_p: \tag{3.5}$$

$$A_{p,m-p}^{(n)} = -\left[\frac{2p-m}{2}\right]_{n;-} \equiv -\left[\frac{2p-m}{2}\right]\Big/\,[n]\,. \tag{3.6}$$

Combining these facts with the results of the KP hierarchy [2], it is natural to think that D-Vir$_{\rm c}^{(+,0)}$ describes some symmetry structure of the KP hierarchy. Now we regard z in (3.1) as the spectral parameter, and expand $\mathcal{L}_m^{(n)}$ in powers of $z\partial_z$. Then we get

$$\mathcal{L}_m^{(n)} = \sum_{j=0}^{\infty}\frac{q^{nm/2} - (-1)^j q^{nm/2}}{q^n - q^{-n}}\frac{(n\lambda)^j}{j!}\sum_{l=0}^{\infty} c_{jl}\ z^{m+l}\partial_z^l \equiv \sum_{l=0}^{\infty}(s_m)_{nl}\ z^{m+l}\partial_z^l \tag{3.7}$$

where c_{jl} is

$$c_{jl} = \sum_{\alpha=1}^{l}\frac{(-1)^{l-\alpha}\ \alpha^j}{(l-\alpha)!\ \alpha!} \quad (j, l \geqslant 1) \qquad c_{l0} = c_{0l} = \delta_{l,0} \tag{3.8}$$

and $\lambda = \ln q\ \ (q^N \neq 1;\ \ \forall N \in \mathbb{Z}_{>0})$. From the above discussions

$$\mathcal{L}_m^{(n)}\ \phi = \left[\sum_{l=0}^{\infty}(s_m)_{nl}\ \ M^l L^{m+l}\right]\ \phi \tag{3.9}$$

leads us to consider the flow as

$$L_m^{(n)} \equiv \sum_{l=0}^{\infty}(s_m)_{nl}\ \ \partial_{m+l,l}\,. \tag{3.10}$$

† This fact was pointed out by a referee.

If we define $\hat{L}$ by replacing ∂ with $\hat{\partial}$ in (3.10), we obtain

$$\hat{L}_m^{(n)}\ \tau = \oint \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}}\ \ \mathcal{L}_m^{(n)}(z) X(z,\zeta)\ \Big|_{z=\zeta}\ \tau$$

$$= \oint \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}}\ \zeta^m\ \frac{q^{nm/2} X(q^n\zeta,\zeta) - q^{-nm/2} X(q^{-n}\zeta,\zeta)}{q^n - q^{-n}}\ \tau \tag{3.11}$$

(cf equation (2.12)).

On the other hand, for the double expansion of $X(z,\zeta)$

$$X(z,\zeta) = \sum_{l=0}^{\infty} \frac{(z-\zeta)^l}{l\ !} \sum_{p=-\infty}^{\infty} z^{-p-l}\ W_p^{(l)} \tag{3.12}$$

we can show that

$$\hat{\partial}_{m+l,l}\ \tau = \frac{1}{l+1} W_m^{(l+1)}\ \tau. \tag{3.13}$$

This means that the coefficient $W_m^{(k)}$ is simply the element of $W_{1+\infty}$. Then

$$\hat{L}_m^{(n)}\ \tau = \sum_{k=1}^{\infty} (\tilde{s}_m)_{n,\ k-1}\ W_m^{(k)}\ \tau \tag{3.14}$$

$((\tilde{s}_m)_{n,k-1} = k^{-1}(s_m)_{n,k-1})$. Now we understand that there is a one-to-one correspondence between $\hat{L}_m^{(n)}$ and $W_m^{(k)}$. At the end of this section, we make sure that $\hat{L}_m^{(n)}$ can be recognized as the action of D-$\mathrm{Vir}_{\mathrm{c}}^{(+,0)}$ on τ. In the Fock representation [2], the function τ is defined as

$$\tau(t,g) = \langle 0|\mathrm{e}^{H(t)} g|0\rangle \qquad g \in GL(\infty) \tag{3.15}$$

where

$$H(t) = \sum_{r=1}^{\infty} H_r t_r \qquad H_r = \sum_{l\in\mathbb{Z}+\frac{1}{2}} :\psi_l \psi_{r-l}^* :\ .$$

The action of the vertex operators is given by

$$X(z,\zeta)\ \tau(t,g) = \langle 0|\mathrm{e}^{H(t)} : \psi(z)\psi^*(\zeta) : g|0\rangle. \tag{3.16}$$

Therefore equation (3.11) becomes

$$\hat{L}_m^{(n)}\ \tau = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}}\ \langle 0|\mathrm{e}^{H(t)} : \big(\mathcal{L}_m^{(n)}\psi(z)\big)\ \ \psi^*(z) : g|0\rangle$$

$$= \langle 0|\mathrm{e}^{H(t)}\left(-\frac{1}{2}\sum_l \left[\frac{2l-m+1}{2}\right]_{n;-} :\psi_l\psi_{m-l}^* :\right) g|0\rangle$$

$$= \langle 0|\mathrm{e}^{H(t)} \hat{\mathcal{L}}_m^{(n)} g|0\rangle. \tag{3.17}$$

In the last line, we use equation (3.5).

It is worthwhile to note that the degree of freedom of the conformal spin k in $W_{1+\infty}$ changes one of the parameters, n, in D-$\mathrm{Vir}_{\mathrm{c}}^{(+,0)}$, which is associated with the difference interval of the new flow. This fact implies that n contains some physical information.

4. Concluding remarks

We have analysed how the simplest D-Virasoro subalgebra, $D\text{-Vir}^{(+,0)}$, can be regarded as the $W_{1+\infty}$ algebra. The implication of this fact is significant. The W-infinity symmetry is considered to be a universal symmetry structure for integrable systems in the sense that such a symmetry often appears in various theories. For example, in [22] it was indicated that the W-infinity algebras provided a unified viewpoint of various integrable hierarchies originated in the KP hierarchy. On the other hand, as mentioned in the introduction, some discretization of the variables preserves the integrability of the system. Therefore $D\text{-Vir}^{(+,0)}$ is not only a deformation of the Virasoro algebra, but is also a discretization of the spectral parameter which preserves the integrability of the KP hierarchy. This point of view should provide useful tools for understanding the structure of integrable systems, such as soliton theories, 2D gravity, etc.

In what follows we make some remarks on the subject.

(i) As mentioned above, we have not used the whole structure of D-Vir. The remaining part must also have a rich structure.

(a) If we consider the $D\text{-Vir}^{(+,r\neq 0)}$ algebra, the fermion is no longer a unique constituent. In this case it is natural to think that the representation is also 'discretized', i.e. we must consider the deformation of the KP hierarchy itself. It would be interesting to know whether or not such a system is still integrable.

(b) Another type of D-Virasoro subalgebra, $D\text{-Vir}^{(-)}$, is also related to the W-infinity algebra. From the fact that the generator of $D\text{-Vir}^{(-,r=1)}$ is realized by ordinary free bosons in (3.5), it is reasonable to guess that it is related to the W_∞ algebra [17].

(c) The representation theory of W-infinity algebras has progressed much recently [23, 24]. The status of $D\text{-Vir}^{(+,0)}$ in the theory should be clarified.

(ii) Even though we have not made use of the fact, it is known that the algebra of Moyal brackets [18] underlies D-Virasoro [7]. Specifically, the latter can be seen to be a subalgebra of Moyal, as detailed in [19–21]. This suggests that the Moyal structure must contain much information on integrability. For example, in [25] it was shown that quantizations on $C^\infty(T^*\mathbb{R})$ generally parametrize realizations of the integrable dynamical system, and the Hamilton form on the phase space was established. In particular, some soliton equations (KdV, Boussinesq, KP) were discussed for the case of the Moyal formalism. In [26], starting with the dispersionless KP hierarchy (d-KP) [27, 28] which is associated with the Poisson algebraic structure, an attempt was made to reconstruct the KP hierarchy as the Moyal-like deformation of d-KP. Therefore these examples support our claim that study of the symmetry with the Moyal structure seems to be of value in obtaining a general view of integrable systems. Moreover, in contrast to the case of q-deformation, which corresponds to the quantization on the Hilbert space, we can proceed with the Moyal-like deformation in parallel with the q-deformation.

(iii) In [29], the $W_{1+\infty}$ algebra was discussed in the context of the Hamilton structure of the soliton system [30]. It would also be interesting to investigate the connection between the result of the present paper and such a structure.

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Appendix

In this appendix, we prove equation (2.12). We first write down the action of the operator $z^m \partial_z^k$ on the wavefunction $\phi(z) = K\mathrm{e}^{\xi(t,z)}$ in terms of pseudo-differential operators:

$$z^m \partial_z^l \phi(z) = K\Gamma^l \partial^m \mathrm{e}^{\xi}. \tag{A.1}$$

After multiplying the adjoint wavefunction $\phi^*(z) = (K^*)^{-1}\mathrm{e}^{-\xi(t,z)}$ from the right on both sides, we integrate along a contour around $z = \infty$:

$$\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \ \left(z^m \partial_z^l \phi(z)\right)\ \phi^*(z) = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \ \left(K\Gamma^l \partial^m \mathrm{e}^{xz}\right)\ \left((K^*)^{-1}\mathrm{e}^{-xz}\right) \tag{A.2}$$

and look at how each side is expressed by using the τ function.

The left-hand side of (A.2)

The wavefunctions can be written as

$$\phi(z,t) = \frac{V(z)\ \tau}{\tau} \qquad \phi^*(z,t) = \frac{V^*(z)\ \tau}{\tau} \tag{A.3}$$

where the vertex operators V and V^* are as defined in section 2. They satisfy the following anticommutation relation:

$$\{V(z), V^*(\zeta)\} = \delta\left(\frac{z}{\zeta}\right). \tag{A.4}$$

The δ function is defined by the formal expansion

$$\delta(z) = \sum_{k\in\mathbb{Z}} z^k. \tag{A.5}$$

If we use (A.4) and the bilinear identity for the wavefunctions

$$\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \ \phi(z,t)\ \phi^*(z,t') = 0 \tag{A.6}$$

we can write the bilinear form $\phi(z)\phi^*(\zeta)$ by means of the vertex operators and the τ function as

$$V(z)V^*(\zeta)\ (\partial \ln \tau).$$

Hence the left-hand side of (A.2) is written as

$$\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \ \left(z^m \partial_z^k V(z)\right)\ V^*(z)\ (\partial \ln \tau)\,. \tag{A.7}$$

The right-hand side of (A.2)

We use the following lemma [12, 13].

Lemma. For any two pseudo-differential operators P and Q

$$\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \ (P\mathrm{e}^{zx})(Q\mathrm{e}^{-zx}) = \mathrm{res}_\partial\ PQ^* \tag{A.8}$$

where $\mathrm{res}_\partial \sum a_k \partial^k = a_{-1}$.

Then the right-hand side of (A.2) becomes

$$\mathrm{res}_\partial\ K\Gamma^l\partial^m K^{-1} = \mathrm{res}_\partial\ M^l L^m = \mathrm{res}_\partial\ (M^l L^m)_- K = -\mathrm{res}_\partial\ \partial_{ml}\phi(z) = -\partial_{ml} w_1\,. \tag{A.9}$$

Since w_1 can be written in terms of the τ function as $w_1 = -\partial \ln\tau$, the right-hand side of (A.9) becomes $\partial_{ml}(\partial \ln\tau)$.

Combining these results with (2.6) and $V(z)V^*(\zeta)\partial = \partial V(z)V^*(\zeta)$, we finally obtain equation (2.12).

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